Solution Sheet 6

1. (i) For a cycle of length 2 we have $(a,b) = (b,a)$ we need only count the number of (unordered) *subsets* of $\{1,2,3,4,5\}$ of size 2. There are $\binom{5}{2} = 10$ of these.

   (ii) For a 3-cycle we have $(a,b,c) = (b,c,a) = (c,a,b)$. So we need count the number of *ordered* 3-tuples, of which there are $5 \times 4 \times 3$ choices, but then divide this by 3. So there are 20 3-cycles.

   (iii) If the permutations fix 5 say, we are counting the number of elements that permute $\{1,2,3,4\}$, which is the same as the cardinality of S_4, i.e. $4! = 24$.

   Only the set of permutations that fix a given element is closed under composition.

2. (i)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & 5 & 10 & 11 & 7 & 4 & 9 & 1 & 2 & 3 & 6 \end{pmatrix}$$
$$= (1,8) \circ (2,5,7,9) \circ (3,10) \circ (4,11,6)\,.$$

   So the order is $\operatorname{lcm}(2,4,2,3) = 12$.

   (ii)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 4 & 6 & 8 & 10 & 5 & 7 & 9 & 11 & 1 & 3 \end{pmatrix}$$
$$\circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 6 & 9 & 1 & 4 & 7 & 10 & 2 & 5 & 8 & 11 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 5 & 11 & 2 & 8 & 7 & 1 & 4 & 10 & 9 & 3 \end{pmatrix}$$
$$= (1,6,7) \circ (2,5,8,4) \circ (3,11) \circ (9,10)\,.$$

   So the order is $\operatorname{lcm}(3,4,2,2) = 12$.

3. ★ Be careful, the cycles in these compositions are not disjoint. You should first write them out as compositions of disjoint cycles.

(i) $(1, 2, 3) \circ (1, 3, 4) \circ (1, 3, 5) = (1, 4, 2, 3, 5)$ which has order 5,

(ii) $(1, 2) \circ (1, 3) \circ (1, 4) \circ (1, 5) = (1, 5, 4, 3, 2)$ which has order 5,

(iii) $(2, 3, 5) \circ (1, 2) \circ (2, 4) \circ (1, 2) = (1, 4) \circ (2, 3, 5)$ which has order 6.

4. a)

$$\begin{aligned}
\sigma_1 &= (1,2,4)\circ(3,5)\,,\\
\sigma_2 &= (2,4,6,5)\,,\\
\sigma_3 &= (1,6,5,4)\circ(2,3)\,,\\
\tau_1 &= (1,7,2)\circ(3,9)\circ(4,8,5,6)\,,\\
\tau_2 &= (2,7)\circ(3,5,8)\circ(4,9,6)\,.
\end{aligned}$$

b) The order of σ_1 is $\operatorname{lcm}(3,2)=6$, of σ_2 is 4, of σ_3 is $\operatorname{lcm}(4,2)=4$, of τ_1 is $\operatorname{lcm}(3,2,4)=12$ and of τ_2 is $\operatorname{lcm}(2,3,3)=6$.

5. You want to find cycles of length $a,b,c,...\geq 2$ with $a+b+c+...=13$ and the largest possible lowest common multiple $\operatorname{lcm}(a,b,c,...)$.

   A search will give $a=7, b=6$, when the maximal order will be 42. Such a permutation would be

$$(1,2,3,4,5,6,7)\circ(8,9,10,11,12,13)\,.$$

6. No. It is not well defined on $\mathbb{Q}$. For instance,

$$\frac{1}{2}*\frac{1}{1}=\frac{2}{3}.$$

   But in $\mathbb{Q}$ we have $\frac{1}{2}=\frac{2}{4}$ yet

$$\frac{2}{4}*\frac{1}{1}=\frac{3}{5}\neq\frac{2}{3}.$$

7. (i) No. $1+1=2$ which is not odd.

   (ii) Yes. If a and b are even integers then $a=2k$ and $b=2\ell$ for some integers k,ℓ. But then $ab=(2k)(2\ell)=2(2k\ell)$ is even.

   (iii) Yes. If a and b are odd integers then $a=2k+1$ and $b=2\ell+1$ for some integers k,ℓ. Thus

$$\begin{aligned}
a\circ b &= a+b-ab\\
&= 2k+1+2\ell+1-(2k+1)(2\ell+1)\\
&= 2(-2k\ell)+1
\end{aligned}$$

   which is odd.

8. (i) Is commutative. Proof: addition on $\mathbb{R}$ is commutative,

Is not associative. Counterexample: $1 * (2 * 3) = 1 * 10 = 22$ while $(1 * 2) * 3 = 18$.

(ii) Is not commutative. Counterexample: $1*-1 = 1$ while $-1*1 = -1$.

Is associative. Proof: $a * (b * c) = a * (b\,|c|) = a\,|b\,|c|| = a\,|b|\,|c|$ and $(a * b) * c = (a\,|b|) * c = a\,|b|\,|c|$.

(iii) Is commutative. Proof: both addition and multiplication are commutative on $\mathbb{R}$.

Is not associative. Counterexample:

$$\begin{aligned} 1 * (2 * 3) &= \frac{1 + (2 * 3)}{1 \times (2 * 3)} = \frac{1 + \frac{2+3}{2\times3}}{\frac{2+3}{2\times3}} = \frac{11}{5}. \\ (1 * 2) * 3 &= \frac{(1 * 2) + 3}{(1 * 2) \times 3} = \frac{\frac{1+2}{2} + 3}{\frac{1+2}{2}} = \frac{9}{3}. \end{aligned}$$

(iv) Is commutative Proof: addition and multiplication are commutative on $\mathbb{Z}$.

Is associative. Proof:

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a\,(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc \\ &= a + b - ab + c - ac - bc + abc \\ &= (a + b - ab) + c - (a + b - ab)\,c \\ &= (a * b) + c - (a * b)\,c \\ &= (a * b) * c. \end{aligned}$$

(v) Is commutative, Proof: $x * y = \max(x, y) = \max(y, x) = y * x$.

Is associative. Proof:

$$\begin{aligned} x * (y * z) &= \max(x, y * z) = \max(x, \max(y, z)) \\ &= \max(x, y, z) = \max(\max(x, y), z) \\ &= \max(x * y, z) = (x * y) * z. \end{aligned}$$

9. (i) Yes. $\max(1, n) = \max(n, 1) = n$ for any $n \geq 1$ and so 1 is the identity.

(ii) No. *If* $e \in \mathbb{Z}$ is an identity, then we can choose an integer $x < e$ and for this integer we find that $x * e = \max(x, e) = e \neq x$.

(iii) Yes. 0. We have seen that this $x * y$ is commutative so we need only examine $x * 0 = x + 0 - x \times 0 = x$.

(iv) Yes. $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

10. Many reasons. Perhaps because subtraction is not associative.

$$1 - (2 - 3) = 2 \quad \text{but} \quad (1 - 2) - 3 = -4.$$

11. a)

$\times_{15}$	1	4	7	13
1	1	4	7	13
4	4	1	13	7
7	7	13	4	1
13	13	7	1	4

$\times_{15}$	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

$\times$	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$D = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) In $(\{1, 4, 7, 13\}, \times_{15})$, the identity is 1 and the inverses are $4^{-1} = 4$, $7^{-1} = 13$ and $13^{-1} = 7$.

In $(\{3,6,9,12\}, \times_{15})$ the identity is 6 and the inverses are $9^{-1} = 9$, $3^{-1} = 12$ and $12^{-1} = 3$.

In the matrix group the identity is A and all matrices are self-inverse.

12. a)

$\times_{28}$	4	8	12	16	20	24
4	16	4	20	8	24	12
8	4	8	12	16	20	24
12	20	12	4	24	16	8
16	8	16	24	4	12	20
20	24	20	16	12	8	4
24	12	24	8	20	4	16

The identity element is 8.

$4^{-1} = 16$, $8^{-1} = 8$, $12^{-1} = 24$, $16^{-1} = 4$, $20^{-1} = 20$ and $24^{-1} = 12$.

b) $(\{4,8,16\}, \times_{28})$ is a closed subset.

$\times_{28}$	4	8	16
4	16	4	8
8	4	8	16
16	8	16	4

$(\{8,20\}, \times_{28})$ is a closed subset:

$\times_{28}$	8	20
8	8	20
20	20	8